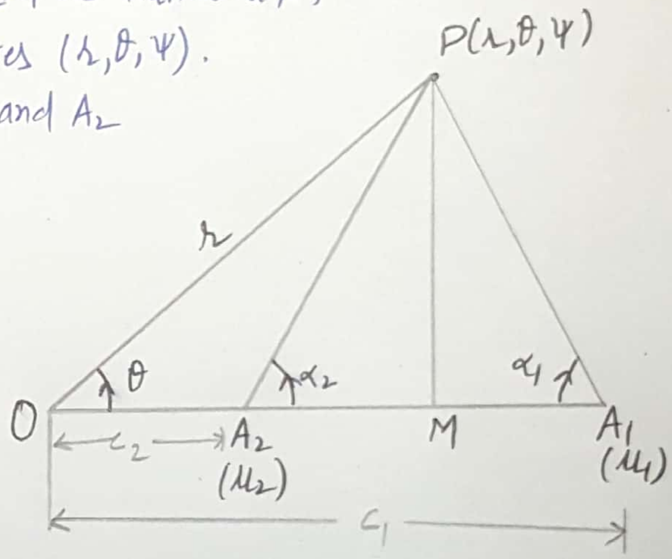
Reference: F. Chorlton
Textbook of Fluid Dynamics.

2. Example: Doublets of strengths μ_1, μ_2 are situated at points A_1, A_2 whose Cartesian coordinates are $(0,0,c_1), (0,0,c_2)$, their axes being directed towards and away from the origin resp. Find the condition that there is no transport of fluid over the surface of the sphere $x^2+y^2+z^2 = c_1c_2$.

Solution: From the figure, with OA_2A_1 as initial line,
Let P has sperical polar coordinates (r,θ,ψ).
Then the axes of the doublet at A_1 and A_2 makes angles α_1, α_2 with A_1P, A_2P.
Thus, the velority potential ϕ at P is



$$\phi = \frac{\mu_2 \cos\alpha_2}{A_2P^2} + \frac{\mu_1 \cos\alpha_1}{A_1P^2} \quad \text{——} (1)$$

It is given that $OA_1 = c_1$ and $OA_2 = c_2$.
then from figure,

$$A_1P^2 = (r^2 - 2rc_1\cos\theta + c_1^2)$$
$$A_2P^2 = (r^2 - 2rc_2\cos\theta + c_2^2) \quad \text{——} (2)$$

$\because \cos\theta = \frac{OM}{OP} = \frac{OM}{r}$

$OM = r\cos\theta$

$$\therefore \cos\alpha_1 = \frac{MA_1}{A_1P} = \frac{OA_1 - OM}{A_1P} = \frac{c_1 - r\cos\theta}{A_1P} \quad \text{——} (3)$$

and
$$\cos\alpha_2 = \frac{MA_2}{A_2P} = \frac{OM - OA_2}{A_2P} = \frac{r\cos\theta - c_2}{A_2P} \quad \text{——} (4)$$

where M is the foot of the perpendicular drawn from P on OA_1.

Using (3) and (4), then (1) reduces to

$$\phi(r,\theta) = \frac{\mu_2(r\cos\theta - c_2)}{A_2P^3} + \frac{\mu_1(c_1 - r\cos\theta)}{A_1P^3} \quad \text{——} (5)$$

Using (2), (5) becomes

$$\phi(r,\theta) = \mu_2 (r\cos\theta - c_2)(r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2} + \mu_1 (c_1 - r\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2}$$

Thus,

$$\frac{\partial\phi}{\partial r} = \mu_2 \left\{ \cos\theta (r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2} - 3(r\cos\theta - c_2)(r - c_2\cos\theta)(r^2 - 2rc_2\cos\theta + c_2^2)^{-5/2} \right\} + \mu_1 \left\{ -\cos\theta (r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2} - 3(c_1 - r\cos\theta)(r - c_1\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-5/2} \right\} \quad \text{——(6)}$$

Since, there is no transport of fluid over the sphere $x^2 + y^2 + z^2 = (\sqrt{c_1c_2})^2$,

we have $\frac{\partial\phi}{\partial r} = 0$ when $r = \sqrt{c_1c_2}$ ——(7)

Hence, using (7), then (6) reduces to

$$\mu_2 \left[\cos\theta \left(c_1c_2 - 2c_2(c_1c_2)^{1/2}\cos\theta + c_2^2\right)^{-3/2} - 3\left((c_1c_2)^{1/2}\cos\theta - c_2\right)\left((c_1c_2)^{1/2} - c_2\cos\theta\right) \times \left(c_1c_2 - 2c_2(c_1c_2)^{1/2}\cos\theta + c_2^2\right)^{-5/2} \right]$$

$$= \mu_1 \left[\cos\theta \left(c_1c_2 - 2c_1(c_1c_2)^{1/2}\cos\theta + c_1^2\right)^{-3/2} + 3\left(c_1 - (c_1c_2)^{1/2}\cos\theta\right)\left((c_1c_2)^{1/2} - c_1\cos\theta\right) \times \left(c_1c_2 - 2c_1(c_1c_2)^{1/2}\cos\theta + c_1^2\right)^{-5/2} \right]$$

On simplification, we obtain

$$\mu_2 c_2^{-3/2} = \mu_1 c_1^{-3/2}$$

or $$\frac{\mu_2}{\mu_1} = \frac{c_2^{3/2}}{c_1^{3/2}} = \left(\frac{c_2}{c_1}\right)^{3/2}$$

or $$\boxed{\frac{\mu_2}{\mu_1} = \left(\frac{c_2}{c_1}\right)^{3/2}}$$

which is the required condition.

4.3 Images in a Rigid infinite plane :-

Suppose a surface S can be drawn in a moving fluid in a such way that there is no transport of fluid across. Let us suppose, S divide the fluid into two regions labelled 1, 2. Then any system of sources, sinks or doublets in 2 is called an image system of the region 1 in S. If we remove the fluid in 2 and replace S by a rigid boundary of the same size and shape, then the flow in 1 is unaltered in accordance with the conditions of the uniqueness theorems. It follows that, if we know the image system for 1 in S, we can solve the problem of flow in 1 against a rigid surface S.

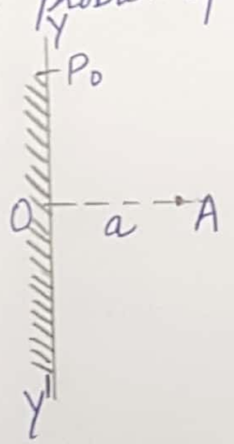


Figure1(i)

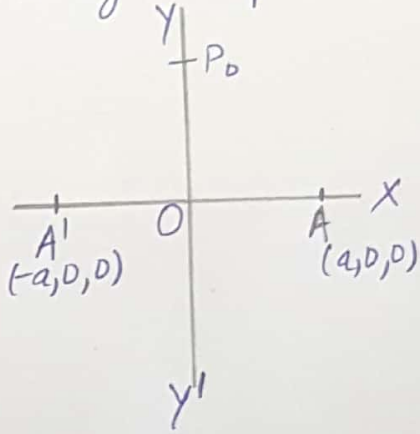


Figure1(ii)

Figure 1(i) shows a simple source of strength m situated at a distance a from an infinite rigid plane YY'. We will show that "the appropriate image system for this is an equal source at A', the optic image of A in the plane."

For this end, Figure 1(ii) in which we have equal sources of strength m at $A(a,0,0)$ and at $A'(-a,0,0)$. Let P_0 be any point on the plane YY' in Figure 1(ii). Then the fluid-velocity at P_0 is

$$= \left(\frac{m}{AP_0^3}\right)\overline{AP_0} + \left(\frac{m}{A'P_0}\right)\overline{A_1P_0}$$

$$= \left(\frac{m}{AP_0^3}\right)(\overline{AP_0} + \overline{A_1P_0}) = \left(\frac{2m}{AP_0^3}\right)\overline{OP_0}$$

$\left\{ \because \overline{AP_0} + \overline{A'P_0} = 2\overline{OP_0} \right.$ since

This shows that at any point P_0 on YY', the fluid flows tangentially to the plane YY'. Thus, there is no transport of fluid across this plane.

Thus, in Figure 1(i), 1(ii), at all corresponding points P_0 on the surfaces YY', $\frac{\partial \phi}{\partial \eta} = 0$, for the region of flow $x \geq 0$.

By the uniqueness theorems, we infer, then that "the image of m at A in YY' in figure 1(i) is at A', the optic image of A in YY'."

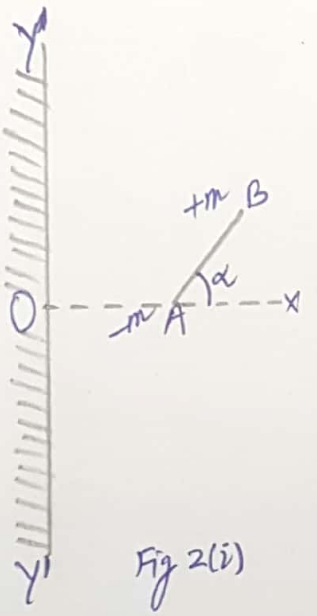


Fig 2(i)

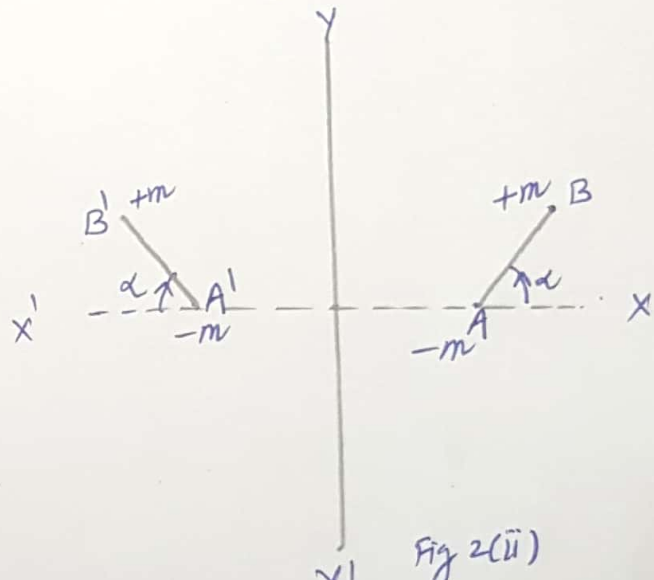


Fig 2(ii)

Now consider a pair of sources $-m$ at A, $+m$ at B, close together and on one side of the rigid plane YY' (Fig 2(i)). The image system is $-m$ at A', m at B', where A', B' are the respective optic images of the points A, B in the plane YY' (Fig 2(ii)).

In the limits it follows, then, that "the image of a doublet in an infinite rigid plane is an equal doublet symmetrically disposed with respect to the plane.

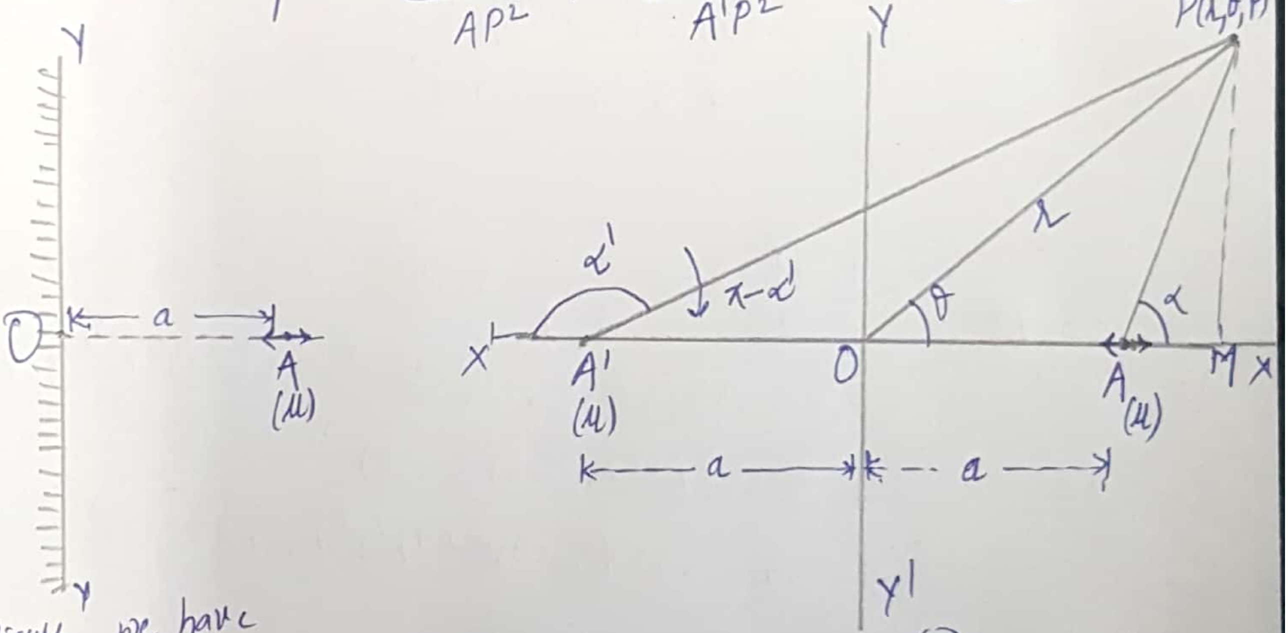
Example:- P-145

A three-dimensional doublet of strength μ whose axis is in the direction $\overline{Ox}$ is distant a from the rigid plane $x=0$ which is the sole boundary of liquid of density ρ, infinite in extent. Find the pressure at a point on the boundary distant r from the doublet given that the pressure at infinity is p_∞. Show that the pressure on the plane is least at a distance $a\sqrt{5}/2$ from the doublet.

Solution:-

Let us suppose YY' be the rigid boundary. Then we know that the appropriate image system of a doublet of strength μ at $A(a,0,0)$ consists of a doublet of the same strength μ at $A'(-a,0,0)$. Then, the velocity potential at P is given by

$$\phi = \frac{\mu\cos\alpha}{AP^2} + \frac{\mu\cos\alpha'}{A'P^2} \quad ----(1)$$



From figure, we have

$$AP^2 = r^2 + a^2 - 2ra\cos\theta \quad ----(2)$$

and

$$A'P^2 = r^2+a^2 - 2ra\cos(\pi-\theta) = r^2+a^2+2ra\cos\theta \quad ----(3)$$

Also

$$\cos\alpha = \frac{AM}{AP} = \frac{OM-OA}{AP} = \frac{r\cos\theta - a}{AP} \quad ----(4)$$

and

$$\cos\alpha' = -\cos(\pi-\alpha') = -\frac{A'M}{A'P} = -\frac{(A'O+OM)}{A'P}$$

$$\therefore \cos\alpha' = -\frac{(a+r\cos\theta)}{A'P} \quad ----(5)$$

where M is the foot of perpendicular drawn from P on OX.

Using (4) and (5), then (1) reduces to

$$\phi = \frac{\mu(r\cos\theta - a)}{AP^3} - \frac{\mu(r\cos\theta + a)}{A'P^3}$$

$$\phi = \frac{\mu(r\cos\theta - a)}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{\mu(r\cos\theta + a)}{(r^2+a^2+2ra\cos\theta)^{3/2}}$$

$$\therefore \phi = \mu\left\{(r\cos\theta - a)(r^2+a^2-2ra\cos\theta)^{-3/2} - (r\cos\theta+a)(r^2+a^2+2ra\cos\theta)^{-3/2}\right\} \quad \text{---} (6)$$

Thus, $$q_r = \frac{-\partial\phi}{\partial r} = -\mu\left\{\cos\theta\,(r^2+a^2-2ra\cos\theta)^{-3/2}\right.$$
$$- 3(r-a\cos\theta)(r\cos\theta - a)(r^2+a^2-2ra\cos\theta)^{-5/2}$$
$$- \cos\theta\,(r^2+a^2+2ra\cos\theta)^{-3/2}$$
$$\left. + 3(r+a\cos\theta)(r\cos\theta - a)(r^2+a^2+2ra\cos\theta)^{-5/2}\right\}$$

using (6)

and $$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\mu}{r}\left\{-r\sin\theta\,(r^2+a^2-2ra\cos\theta)^{-3/2}\right.$$
$$- 3ra\sin\theta\,(r\cos\theta - a)(r^2+a^2-2ra\cos\theta)^{-5/2}$$
$$+ r\sin\theta\,(r^2+a^2+2ra\cos\theta)^{-3/2}$$
$$\left. - 3ra\sin\theta\,(r\cos\theta + a)(r^2+a^2+2ra\cos\theta)^{-5/2}\right\}$$

When $\theta = \pi/2$, then we have

$$q_r = -6\mu ra(r^2+a^2)^{-5/2}$$

and $$q_\theta = 0, \quad q_\psi = 0$$

Along the streamline through $(r, \pi/2, \psi)$, Bernoulli's equation is

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{p_\infty}{\rho} \quad \text{----} (7)$$

Here $q = q_r = -6\mu r a(r^2+a^2)^{-5/2}$

Thus (7) gives $\frac{p}{\rho} + 18\mu^2 a^2 r^2 (r^2+a^2)^{-5} = \frac{p_\infty}{\rho}$

$$\therefore \quad p = p_\infty - 18\mu^2 a^2 r^2 \rho (r^2+a^2)^{-5} \quad \text{---} (8)$$

From (8), $\frac{dp}{dr} = -18\mu^2 a^2 \rho \left\{ 2r(r^2+a^2)^{-5} - 5r^2(r^2+a^2)^{-6} \cdot 2r \right\}$

$$= -18\mu^2 a^2 \rho \cdot 2r (r^2+a^2)^{-5} \left\{ 1 - \frac{5r^2}{r^2+a^2} \right\}$$

$$= -36\mu^2 a^2 \rho r (r^2+a^2)^{-5} \cdot \left\{ \frac{a^2-4r^2}{r^2+a^2} \right\}$$

Thus. $\frac{dp}{dr} = 36\mu^2 a^2 \rho r (r^2+a^2)^{-6} \cdot (4r^2-a^2) \quad \text{----} (9)$

For Maximum and Minimum values of p, we must have

$\frac{dp}{dr} = 0$ or $36\mu^2 a^2 \rho r (r^2+a^2)^{-6} (4r^2-a^2) = 0$ From (9)

it gives $r = 0,\ r = \frac{a}{2}, -\frac{a}{2}$

Now, from (9), we will get

$$\frac{dp}{dr} = 144\mu^2 a^2 \rho r (r^2+a^2)^{-6} \left(r^2 - \frac{a^2}{4}\right)$$

Hence, $\frac{dp}{dr} = 144\mu^2 a^2 \rho r (r^2+a^2)^{-6} \left(r - \frac{a}{2}\right)\left(r + \frac{a}{2}\right)$.

$\therefore$ when r is slightly less than $\frac{a}{2}$, $\frac{dp}{dr}$ is negative.

and when r is slightly greater than $\frac{a}{2}$, $\frac{dp}{dr}$ is positive.

Hence p must be minimum at $r = \frac{a}{2}$ on the plane.

i.e at a distance $\left(a^2 + \frac{a^2}{4}\right)^{1/2}$ i.e $a\sqrt{5}/2$ from the doublet.

Hence results.

4.4. Images in Solid Spheres:

P-147.

Here, we establish some properties of harmonic functions specified by a system of spherical polar coordinates (r, θ, ψ). Such functions satisfy the spherical polar form of Laplace's equation, viz.

$$\nabla^2 \phi \equiv \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \psi}\right)\right\} = 0 \quad \text{--- (1)}$$

I Theorem: If $r^n S_n(\theta, \psi)$ is a harmonic function, so also is $r^{-(n+1)} S_n(\theta, \psi)$.

Proof: we have,

Since $\phi = r^n S_n(\theta, \psi)$ satisfies (1).

so that

$$\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \psi}\right) = 0$$

$$\therefore \quad \frac{\partial}{\partial r}\left(n r^{n+1} S_n \sin\theta\right) + \frac{\partial}{\partial \theta}\left(r^n \sin\theta \frac{\partial S_n}{\partial \theta}\right) + \operatorname{cosec}\theta \frac{\partial}{\partial \psi}\left(r^n \frac{\partial S_n}{\partial \psi}\right) = 0$$

or

$$n(n+1) S_n \sin\theta + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_n}{\partial \theta}\right) + \operatorname{cosec}\theta \frac{\partial^2 S_n}{\partial \psi^2} = 0 \quad \text{--- (2)}$$

Also, we take $\Phi = r^{-(n+1)} S_n(\theta, \psi)$ in (1), we get

$$r^2 \sin\theta \, \nabla^2 \Phi = \frac{\partial}{\partial r}\left\{-(n+1) r^{-n} \sin\theta \, S_n\right\} + r^{-(n+1)} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_n}{\partial \theta}\right) + \operatorname{cosec}\theta \, r^{-(n+1)} \frac{\partial^2 S_n}{\partial \psi^2}$$

$$= r^{-(n+1)}\left\{(n+1) n \, S_n \sin\theta + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_n}{\partial \theta}\right) + \operatorname{cosec}\theta \frac{\partial^2 S_n}{\partial \psi^2}\right\}$$

$$r^2 \sin\theta \, \nabla^2 \Phi = 0, \quad \text{using (2).}$$

it follows that $\Phi = r^{-(n+1)} S_n(\theta, \psi)$ is also a harmonic function.

Hence Proved.

II Theorem:- if $\phi(r,\theta,\psi)$ has an expansion of the form

$$\phi(r,\theta,\psi) = \sum_{n=0}^{\infty} \alpha_n r^n S_n(\theta,\psi).$$

the series on the right being uniformly convergent with respect to r, then for constant λ,

$$\frac{1}{r^\lambda}\int_0^r R^{\lambda-1}\phi(R,\theta,\psi)\,dR = \sum_{n=0}^{\infty} \frac{\alpha_n r^n S_n(\theta,\psi)}{n+\lambda}$$

Proof. We have

$$R^{\lambda-1}\phi(R,\theta,\psi) = \sum_{n=0}^{\infty} \alpha_n R^{n+\lambda-1} S_n(\theta,\psi).$$

Hence.

$$\int_0^r R^{\lambda-1}\phi(R,\theta,\psi)\,dR = \sum_{n=0}^{\infty} \frac{\alpha_n r^{n+\lambda} S_n(\theta,\psi)}{n+\lambda}$$

whence the result follows.